

INSERTION AND ELIMINATION LIE ALGEBRA: THE LADDER CASE.

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ABSTRACT. We prove that the insertion-elimination Lie algebra of Feynman graphs in the ladder case has a natural interpretation in terms of a certain algebra of infinite dimensional matrices. We study some aspects of its representation theory and we will discuss some relations with the representation of the Heisenberg algebra.

1. INTRODUCTION

In the last few years perturbative QFT had been shown to have a rich algebraic structure [7] and deep relations with apparently unrelated sectors of mathematics like non-commutative geometry and Riemann-Hilbert like problems[2, 3]. Such extraordinary relations can be resumed, to some extent, by the existence of a commutative, non co-commutative Hopf algebra \mathcal{H} defined on the set of Feynman diagrams. In what follows we will discuss some first relations of perturbative QFT with the representation theory of Lie algebras. In fact the Hopf algebra \mathcal{H} is isomorphic via the celebrated Milnor-Moore theorem [8] to the dual of the universal enveloping algebra of a Lie algebra \mathcal{L} . The Lie algebra \mathcal{L} can be realized as derivations of the Hopf algebra \mathcal{H} and it has been shown in [4] to have two natural representations whose action on \mathcal{H} is given by elimination and insertion operators. Here, the goal of the authors is to study the simplest case, i.e the case of the cocommutative sub Hopf algebra of ladder graphs. In this version the Hopf algebra is reduced to a polynomial algebra freely generated by infinitely many generators and insertion and elimination are performed by increasing or decreasing the degree of the generators of such

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algebra. Even in this simplified context the Lie algebra introduced in [4] has non-trivial features.

A further motivation for the study of this Lie algebra comes from [1]. There, a sub Hopf algebra of graphs was studied which is non-cocommutative and contains the one considered here. The Dyson-Schwinger equation for that Hopf algebra was solved by turning the renormalization group into a propagator-coupling duality. The crucial ingredient was the understanding how the elimination Lie algebra (called "befooting" in [1]) commutes with the generator of the equation of motion, the Hochschild one-cocycle B_+ , and how it commutes with the derivation $S \star Y = m \circ (S \otimes Y) \circ \Delta$ (terminology as in that paper). We can easily rewrite the solution of any DSE in the form

$$X = r + \sum_{\text{res}(\Gamma)=r} g^{|\Gamma|} Z_{\Gamma,r}(r),$$

where the residue r specifies the external quantum numbers of the graph and the sum is over all graphs with external legs specified by these quantum numbers. Following [1], we then need to understand the commutators $[Z_{[r,\Gamma]}, Z_{[\Gamma,r]}]$ and $[Z_{[r,\Gamma]}, S \star Y]$ to solve the DSE in accordance with the RG.

We will express below the derivation $S \star Y$ acting on the Hopf algebra (of ladder graphs) in terms of generators of the insertion and elimination Lie algebra, and thus show that the methods of [1] can be formulated entirely in that Lie algebra.

2. THE LIE ALGEBRA \mathcal{L}_L

Let us start recalling the following theorem where we refer for notation and symbols to [4]:

Theorem 2.1. *For all 1-PI graphs Γ_i , s.t. $\text{res}(\Gamma_1)=\text{res}(\Gamma_2)=\text{res}(\Gamma_3)=\text{res}(\Gamma_4)$, the bracket*

$$\begin{aligned} [Z_{[\Gamma_1,\Gamma_2]}, Z_{[\Gamma_3,\Gamma_4]}] &= Z_{[\overline{Z_{[\Gamma_1,\Gamma_2]} \times \delta_{\Gamma_3}}, \Gamma_4]} - Z_{[\Gamma_3, \overline{Z_{[\Gamma_2,\Gamma_1]} \times \delta_{\Gamma_4}}]} - Z_{[\overline{Z_{[\Gamma_3,\Gamma_4]} \times \delta_{\Gamma_1}}, \Gamma_2]} \\ &\quad + Z_{[\Gamma_1, \overline{Z_{[\Gamma_4,\Gamma_3]} \times \delta_{\Gamma_2}}]} - \delta_{\Gamma_2,\Gamma_3} Z_{[\Gamma_1,\Gamma_4]} + \delta_{\Gamma_1,\Gamma_4} Z_{[\Gamma_3,\Gamma_2]}, \end{aligned} \quad (1)$$

defines a Lie algebra of derivations acting on the Hopf algebra \mathcal{H}_{FG} via:

$$Z_{[\Gamma_1,\Gamma_2]} \times \delta_X = \sum_I \langle Z_{\Gamma_2}^+, \delta_{X'_{(i)}} \rangle \delta_{X''_{(i)} *_{G_i} \Gamma_1},$$

where the G_i are normalized gluing data.

We need to translate the formula (1) to the ladder case. We start with the following:

Definition 2.2. $\mathcal{L}_L = \text{span}_{\mathbb{C}} \{Z_{n,m}; n, m \geq 0\}$

This is obvious if we identify the n -loop ladder graph Γ_n in the Hopf algebra with a Hopf algebra element δ_n , n being a non-negative integer, and $Z_{[\Gamma_n, \Gamma_m]}$ with $Z_{n,m}$. Here, any subclass of n -loop graphs Γ_n such that

$$\Delta(\Gamma_n) = \Gamma_n \otimes 1 + 1 \otimes \Gamma_n + \sum_{j=1}^{n-1} \Gamma_j \otimes \Gamma_{n-j}$$

can serve as an example of a Hopf sub algebra of ladder graphs.

Then, the theorem 2.1 becomes:

Theorem 2.3. \mathcal{L}_L is a Lie algebra with commutator given by the following formula:

$$\begin{aligned} [Z_{n,m}, Z_{l,s}] = & \Theta(l-m)Z_{l-m+n,s} - \Theta(s-n)Z_{l,s-n+m} \\ & - \Theta(n-s)Z_{n-s+l,m} + \Theta(m-l)Z_{n,m-l+s} \\ & - \delta_{m,l}Z_{n,s} + \delta_{n,s}Z_{l,m}, \end{aligned} \quad (2)$$

where:

$$\begin{cases} \Theta(l-m) = 0 & \text{if } l < m, \\ \Theta(l-m) = 1 & \text{if } l \geq m \end{cases} \quad (3)$$

and

$$\begin{cases} \delta_{n,m} = 1 & \text{if } m = n, \\ \delta_{n,m} = 0 & \text{if } n \neq m. \end{cases} \quad (4)$$

Let us now introduce some natural module for the Lie algebra \mathcal{L}_L .

Definition 2.4.

$$\mathcal{S} = \bigoplus_{n \geq 0} \mathbb{C}t_n = \mathbb{C}[t_0, t_1, t_2, t_3, \dots].$$

We will assign a degree equal to k to the generator t_k for each $k \geq 0$.

\mathcal{L}_L acts on \mathcal{S} via the following:

$$\begin{aligned} Z_{n,m}t_k &= 0 \text{ if } m > k, \\ Z_{n,m}t_k &= t_{k-m+n} \text{ if } m \leq k. \end{aligned} \tag{5}$$

Remark 2.5. \mathcal{S} is a polynomial algebra with infinite many generators graded as in definition 2.4. The product of two polynomials x_i, x_j will be denoted by concatenation:

$$x_i \otimes x_j \longrightarrow x_i x_j;$$

in particular $\deg(t_i t_j) = i + j$ and y is a unit for \mathcal{S} with respect to this product if and only if it is a scalar. Beside this algebraic structure we need to define on \mathcal{S} the following product:

$$\star : \mathcal{S} \otimes \mathcal{S} \longrightarrow \mathcal{S}$$

$$\star(t_n \otimes t_m) \mapsto t_{n+m}.$$

With respect to this product \mathcal{S} becomes a standard polynomial algebra in one generator with the usual grading:

$$\deg(t_k) = k, \quad \deg(t_k \star t_l) = \deg(t_{k+l}) = l + k.$$

In particular $y \in \mathcal{S}$ is a unit with respect to the \star -product if and only if $y \in \mathbb{C}t_0$.

In what follows we will call \mathcal{S} the standard representation for the Lie algebra \mathcal{L}_L .

We recall that a Lie algebra \mathfrak{g} is called \mathbb{Z} -graded if:

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}.$$

In the decomposition $\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$, the components \mathfrak{g}_n are called homogeneous of degree equal to n .

Proposition 2.6. *The Lie algebra \mathcal{L}_L is \mathbb{Z} -graded,*

$$\mathcal{L}_L = \bigoplus_{n \in \mathbb{Z}} l_n,$$

where the homogeneous components of degree n are:

$$l_n = \text{span}_{\mathbb{C}}\{Z_{k,m}; \ k - m = n\}.$$

PROOF We need to prove that if $Z_{n,m} \in l_i$ and $Z_{l,s} \in l_j$ than:

$$[Z_{n,m}, Z_{l,s}] \in l_{i+j}.$$

This follows by direct computation using (2). \square

From the \mathbb{Z} -grading it follows that \mathcal{L}_L has the following decomposition:

$$\mathcal{L}_L = L^+ \oplus L^0 \oplus L^-;$$

where $L^+ = \oplus_{n>0} l_n$, $L^- = \oplus_{n<0} l_n$ and $L^0 = l_0$.

We have that:

Corollary 2.7. L^+ , L^- and L^0 are sub Lie algebras of \mathcal{L}_L . Moreover L^0 is commutative and fulfills the following:

$$[L^+, L^0] \subseteq L^+, \quad [L^-, L^0] \subseteq L^-.$$

PROOF It follows by direct computation using (2). \square

We thus have the following:

Proposition 2.8. *In terms of the standard representation we have:*

given $n, m \geq 0$, $Z_{n,m}$ is a matrix in which the only non-zero entries are all equal to one and are located on the $n - m$ lower diagonal if $n > m$, or on the $m - n$ upper diagonal if $m > n$. More precisely given $n > m \geq 0$ and $k > 0$:

a) $Z_{n-m,0}$ is a matrix in which the entries of the $n - m$ (lower) diagonal are all equal to 1;

b) $Z_{n-m+k,k}$ is a matrix having zeros on the first $k - 1$ entries of the $n - m$ (lower) diagonal and all the other entries of such diagonal equal to one.

a) and b) hold for the matrices $Z_{0,n-m}$ and $Z_{k,n-m+k}$ substituting, in the previous statements, lower with upper.

PROOF It follows directly from (5). \square The following claim is now evident:

Lemma 2.9. *Given n, m as in the lemma (2.8), we have that: if $n - m > 0$ $Z_{n,m} \in L^+$, if $n - m < 0$ $Z_{n,m} \in L^-$ and if $n=m$ $Z_{n,m} \in L^0$.*

We recall that given a graded Lie algebra $\mathfrak{g} = \oplus_{n \in \mathbb{Z}} \mathfrak{g}_n$, a vector space V is a graded \mathfrak{g} -module if $V = \oplus_{n \in \mathbb{Z}} V_n$ and $g_j V_i \subseteq V_{i+j}$, $\forall i, j \in \mathbb{Z}$. The \mathfrak{g} -module V will be said to be of finite type if $\dim V_j < \infty$, $\forall j \in \mathbb{Z}$. So we have:

Proposition 2.10. \mathcal{S} is a graded \mathcal{L}_L -module of finite type.

PROOF It follows from direct computation; given any element $Z_{n,m} \in l_i$ with $i = n - m$:

$$Z_{n,m}t_k = t_{k-m+n} \in l_{k+i}$$

from (5). □

We also recall that a highest weight (h.w) \mathfrak{g} -module of highest weight $\alpha \in \mathfrak{g}_0^*$ is a \mathbb{Z} -graded \mathfrak{g} -module $V(\alpha) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_n$ such that the following properties hold:

- a) $V_0 = \mathbb{C}w_\alpha$, where w_α is a vector not equal to zero;
- b) $hw_\alpha = \alpha(h)w_\alpha$, for all $h \in \mathfrak{g}_0$;
- c) $\mathfrak{g}_-w_\alpha = 0$;
- d) $\mathcal{U}(\mathfrak{g}_+)w_\alpha = V(\alpha)$,

where $\mathfrak{g}_+ = \bigoplus_{n>0} \mathfrak{g}_n$, $\mathfrak{g}_- = \bigoplus_{n<0} \mathfrak{g}_n$ and with $\mathcal{U}(\mathfrak{g})$ we indicated the universal enveloping algebra of \mathfrak{g} . The vector w_α is called highest weight vector (h.w.v) and any vector v such that $\mathfrak{g}_-v = 0$ is called singular vector. Moreover we have that the module $V(\alpha)$ is irreducible if and only if every singular vector is a multiple of the h.w.vector. Now we can state the following:

Proposition 2.11. *\mathcal{S} is an irreducible h.w. module for the Lie algebra \mathcal{L}_L with h.w. vector $w_\alpha = t_0$.*

PROOF It follows directly by the definitions. □

3. CLASSICAL LIE ALGEBRAS AND THE LIE ALGEBRAS \mathcal{L}_L .

In what follows we will investigate some relations of the Lie algebra \mathcal{L}_L with some (infinite dimensional) classical Lie algebras. In particular we will give a complete description of $\mathfrak{sl}_+(\infty)$ in terms of \mathcal{L}_L . Let's start with the following:

Definition 3.1. [5, 6]

$$\mathfrak{gl}(\infty) = \{E_{i,j} : i, j \in \mathbb{Z} : [E_{i,j}, E_{n,m}] = E_{i,m}\delta_{j,n} - E_{n,j}\delta_{m,i}\};$$

and

$$\mathfrak{gl}_+(\infty) = \{E_{i,j} : i, j \geq 0 : [E_{i,j}, E_{n,m}] = E_{i,m}\delta_{j,n} - E_{n,j}\delta_{m,i}\}.$$

Now we can state the following:

Proposition 3.2. *We have an embedding of Lie algebras:*

$$\phi : \mathfrak{gl}_+(\infty) \longrightarrow \mathcal{L}_L. \quad (6)$$

PROOF Define $\phi(E_{i,j}) = Z_{i,j} - Z_{i+1,j+1}$ for each $i, j \geq 0$.

Now it suffices to show that ϕ is morphism of Lie algebras, i.e that:

$$[Z_{i,j} - Z_{i+1,j+1}, Z_{n,m} - Z_{n+1,m+1}] = (Z_{i,m} - Z_{i+1,m+1})\delta_{j,n} - (Z_{n,j} - Z_{n+1,j+1})\delta_{m,i}.$$

This follows directly from the definitions. \square

Definition 3.3. [5, 6]

$$\mathfrak{sl}_+(\infty) = \{A \in \mathfrak{gl}_+(\infty) : \text{tr} A = 0\}$$

We recall now that a set of generators for $\mathfrak{sl}_+(\infty)$ is given by:

$$\begin{cases} E_{i,j} & i < j, \\ E_{i,j} & i > j, \\ E_{i,i} - E_{l,l} & \text{with } i \neq l. \end{cases} \quad (7)$$

From (7) we get the following (standard) triangular decomposition:

$$\mathfrak{sl}_+(\infty) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-; \quad (8)$$

where

$\mathfrak{n}_+ = \text{span}_{\mathbb{C}}\{E_{i,j}; j > i\}$; $\mathfrak{n}_- = \text{span}_{\mathbb{C}}\{E_{i,j}; i > j\}$; $\mathfrak{h} = \text{span}_{\mathbb{C}}\{E_{i,i} - E_{l,l}; i \neq l\}$.

Let us now introduce Chevalley's generators and co-roots for the Lie algebra $\mathfrak{sl}_+(\infty)$ ([5]): The Chevalley's generators are:

$$e_i = E_{i,i+1} \quad f_i = E_{i+1,i} \quad \forall i \in \mathbb{Z}_{\geq 0}$$

and

$$\check{\Pi} = \{E_{i,i} - E_{i+1,i+1}; i \in \mathbb{Z}_{\geq 0}\}$$

is the set of simple co-roots.

We can now write the Chevalley's [5] generators and a co-root system for $\mathfrak{sl}_+(\infty)$ in terms of the generators $Z_{n,m}$ of the Lie algebra \mathcal{L}_L :

Lemma 3.4. *The Chevalley generators for $\mathfrak{sl}_+(\infty)$, $\{e_i, f_i; i \geq 0\}$ can be written in terms of the generators $Z_{n,m}$ in the following way:*

$$\begin{cases} f_i &= Z_{i+1,i} - Z_{i+2,i+1}, \\ e_i &= Z_{i,i+1} - Z_{i+1,i+2}. \end{cases} \quad (9)$$

If we define:

$$\check{\alpha}_i = Z_{i,i} - 2Z_{i+1,i+1} + Z_{i+2,i+2};$$

then $\{\check{\alpha}_i; i \geq 0\}$ is a system of positive simple co-roots for the Lie algebra $\mathfrak{sl}_+(\infty)$.

PROOF It follows from the definition of Chevalley generators, positive simple roots and from the embedding of $\mathfrak{gl}_+(\infty)$ in \mathcal{L}_L defined in (6). \square

Remark 3.5. [5] *The root system for $\mathfrak{sl}_+(\infty)$ is described in the following way; we remember that we have a triangular decomposition for $\mathfrak{gl}_+(\infty)$ analogous to (8):*

$$\mathfrak{gl}_+(\infty) = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where now:

$$\mathfrak{h} = \text{span}_{\mathbb{C}}\{E_{i,i}; i \geq 0\}.$$

Define now:

$$\mathfrak{h}^* = \{\mathbb{C} - \text{linear functions on } \mathfrak{h}\} = \text{span}_{\mathbb{C}}\{\epsilon_i; i \geq 0\},$$

where

$$\epsilon_i(E_{j,j}) = \delta_{i,j},$$

or in terms of $Z_{n,m}$ generators:

$$\epsilon_i(Z_{j,j} - Z_{j+1,j+1}) = \delta_{i,j}.$$

From this description follows immediately:

The root system for $\mathfrak{sl}_+(\infty)$ is given by:

- i) $\Delta = \text{span}_{\mathbb{C}}\{\epsilon_i - \epsilon_j; i \neq j\};$
- ii) $\Delta^+ = \text{span}_{\mathbb{C}}\{\epsilon_i - \epsilon_j; i, j \geq 0; i < j\};$
- iii) $\Pi = \text{span}_{\mathbb{C}}\{\epsilon_i - \epsilon_{i+1}; i \neq 0\};$

where Δ , Δ^+ and Π are respectively called the set of roots, the set of positive roots and the set of simple positive roots.

3.1. The Chevalley's involution on \mathcal{L}_L . We define now an involution on \mathcal{L}_L whose restriction to $\mathfrak{sl}_+(\infty)$ gives us the Chevalley's involution. Let us start with:

Definition 3.6.

$$\begin{aligned} C : \mathcal{L} &\longrightarrow \mathcal{L} \\ Z_{n,m} &\longmapsto -Z_{m,n} \end{aligned}$$

for each $n, m \geq 0$.

We have the following:

Proposition 3.7. *C is an homomorphism of Lie algebras, i.e:*

$$C([Z_{n,m}, Z_{l,s}]) = [C(Z_{n,m}), C(Z_{l,s})], \quad (10)$$

$\forall n, m, l, s \geq 0$.

PROOF It follows applying the (2) to both sides of (10):

$$\begin{aligned} \text{RHS} &= [Z_{m,n}, Z_{s,l}] = \Theta(s-n)Z_{s-n+m,l} - \Theta(l-m)Z_{s,l-m+n} - \\ &\quad \Theta(m-l)Z_{m-l+s,n} + \Theta(n-s)Z_{m,n-s+l} - \delta_{n,s}Z_{m,l} + \delta_{m,l}Z_{s,n}, \end{aligned} \quad (11)$$

$$\begin{aligned} \text{LHS} &= C([Z_{n,m}, Z_{l,s}]) = -\Theta(l-m)Z_{s,l-m+n} + \Theta(s-n)Z_{s-n+m,l} + \\ &\quad \Theta(n-s)Z_{m,n-s+l} - \Theta(m-l)Z_{m-l+s,n} - \delta_{n,s}Z_{m,l} + \delta_{m,l}Z_{s,n}. \end{aligned} \quad (12)$$

□

We recall that:

Definition 3.8. [5] *Given a Lie algebra \mathfrak{g} , a set of its Chevalley's generators $\{f_i, e_i; i \geq 0\}$ and a system of simple positive co-roots $\{\check{\alpha}_i, i \geq 0\}$, the map:*

$$\omega : \mathfrak{g} \longrightarrow \mathfrak{g}$$

defined by:

$$\omega(f_i) = -e_i, \quad \omega(e_i) = -f_i, \quad \omega(\check{\alpha}_i) = -\check{\alpha}_i,$$

is called Chevalley's involution.

Proposition 3.9. *The restriction of the map C to $\mathfrak{sl}_+(\infty)$ is the Chevalley's involution.*

PROOF It is obvious from the definitions:

$$\begin{aligned} C(f_i) &= C(Z_{i+1,i} - Z_{i+2,i+1}) = -Z_{i,i+1} + Z_{i+1,i+2} = e_i, \\ C(\check{\alpha}_i) &= C(Z_{i,i} - 2Z_{i+1,i+1} + Z_{i+2,i+2}) = -Z_{i,i} + 2Z_{i+1,i+1} - Z_{i+2,i+2} = \check{\alpha}_i. \end{aligned}$$

□

4. l^+ , l^- AND THEIR CENTRAL EXTENSIONS.

In this section we are going to consider the relation between the Lie algebra \mathcal{L}_L and the Heisenberg algebra. We will introduce two sub-algebras of \mathcal{L}_L that play the role of "shift" operators for the standard module \mathcal{S} . Let us start with the following:

Definition 4.1. *Define:*

$$l^+ = \text{span}_{\mathbb{C}}\{Z_{n,0}; n \geq 0\} \quad (13)$$

and

$$l^- = \text{span}_{\mathbb{C}}\{Z_{0,n}; n \geq 0\}. \quad (14)$$

We have:

Lemma 4.2. 1)

$$[l^+, l^+] = 0, \quad [l^-, l^-] = 0;$$

2) l^+ acts as algebra of shift operators on \mathcal{S} and l^- as algebra of quasi-shift operators, i.e $\forall n, k > 0$ $Z_{0,n}$ will eliminate any $t_k \in \mathcal{S}$ if $k < n$ and will shift such element to the element t_{k-n} if $n < k$.

PROOF a) follows applying (2) to elements in l^+ and l^- .

b) follows trivially from the definition of the \mathcal{L}_L action on \mathcal{S} . \square

Note that the commutativity of these two algebras stems from the cocommutativity of the ladder Hopf algebra \mathcal{H}_L . The general case provides non-commutative Lie algebras for insertion as well as elimination [4].

Let us now define two other commutative Lie algebras.

Definition 4.3. Let $\{Z_n^+; n \in \mathbb{Z}\}$ and $\{Z_n^-; n \in \mathbb{Z}\}$ two sets of symbols. Let us define:

$$l^{\check{+}} = \text{span}_{\mathbb{C}}\{Z_n^+; n \in \mathbb{Z}\}$$

and

$$l^{\check{-}} = \text{span}_{\mathbb{C}}\{Z_n^-; n \in \mathbb{Z}\}$$

and let us introduce the canonical isomorphism:

$$d : l^{\check{+}} \longrightarrow l^{\check{-}}$$

defined on the generators by:

$$d(Z_n^+) = Z_n^- \quad n \in \mathbb{Z}$$

Now define the following maps:

$$\mathfrak{a}^+ : l^+ \longrightarrow l^{\check{+}}$$

$$Z_{2n,0} \mapsto Z_n^+ \quad \forall n \geq 0$$

$$Z_{2n-1,0} \mapsto Z_{-n}^+ \quad \forall n \geq 1$$

and

$$\mathfrak{a}^- : l^- \longrightarrow l^{\check{-}}$$

$$Z_{0,2n} \mapsto Z_n^- \quad \forall n \geq 0$$

$$Z_{0,2n-1} \mapsto Z_{-n}^- \quad \forall n \geq 1$$

Now we have the following:

Lemma 4.4. \mathfrak{a}^+ and \mathfrak{a}^- are isomorphisms of Lie algebras compatible with the involution C .

PROOF It follows trivially from the definitions of C , d and from the commutativity of the Lie algebras l^\pm and $l^{\check{\pm}}$. \square

It is a well known result that:

Proposition 4.5. [5, 6]

$$H^2(l^{\check{\pm}}, \mathbb{C}) \neq 0$$

PROOF Let's focus on $l^{\check{+}}$ case (the case $l^{\check{-}}$ is completely analogous). Define the following bilinear map:

$$c^+ : l^{\check{+}} \otimes l^{\check{+}} \longrightarrow \mathbb{C}$$

$$(Z_n^+ \otimes Z_m^+) \mapsto n\delta_{n,-m}.$$

c^+ clearly satisfies the following:

$$c^+(Z_n^+, Z_m^+) = -c^+(Z_m^+, Z_n^+) \text{ and}$$

$$c^+\left([Z_n^+, Z_m^+], Z_k^+\right) + \text{cyclic permutations} = 0;$$

i.e c^+ is a non trivial two co-cycle. \square

Remark 4.6. [5, 6] *The co-cycle gives rise to a central extension:*

$$0 \rightarrow \mathbb{C}\mathcal{C} \rightarrow \mathcal{H}^+ \xrightarrow{p} \check{l}^+ \rightarrow 0, \quad (15)$$

where $\mathcal{H}^+ = \text{span}_{\mathbb{C}}\{Z_n^+, \mathcal{C}\}_{n \in \mathbb{Z}}$ and relations:

$$[Z_n^+, Z_m^+] = n\delta_{n,-m}\mathcal{C} \quad [Z_n^+, \mathcal{C}] = 0.$$

\mathcal{H}^+ is called Heisenberg Lie algebra; the "–" case is completely analogous: the co-cycle c^- will give us the central extension \mathcal{H}^- with $\mathcal{H}^- \cong \mathcal{H}^+$.

Let us now consider a \mathcal{H} -module V (in what follows we will drop the \pm sign) on which Z_0 acts as multiplication operator. Let us define the following operators:

$$L_0 = \frac{\mu^2 + \lambda^2}{2} + \sum_{n>0} Z_{-n}Z_n,$$

and

$$L_n = \frac{1}{2} \sum_{j \in \mathbb{Z}} Z_{-j}Z_{j+n} + i\lambda n Z_n \quad \forall n \in \mathbb{Z}^* \quad \lambda \in \mathbb{C}.$$

Then we have that the L_n 's defined above fulfill the Virasoro commutation relations:

$$[L_n, L_m] = (n-m)L_{n+m} + \delta_{n,-m} \frac{(n^3 - n)}{12} (1 + 12\lambda^2) Id_V,$$

where $\lambda \in \mathbb{C}$ gives the central charge of the Virasoro algebra.

It is now time to turn to the derivation $S \star Y$, expressed in generators $Z_{0,n}, Z_{m,0}$ and their commutators.

5. THE DERIVATION $S \star Y$

Let us now turn to an expression of the derivation $S \star Y$ in terms of the generators of the insertion elimination Lie algebra. To that end, let Γ_m correspond to the m -loop ladder graph as a generator in the sub Hopf algebra \mathcal{H}_L of ladder graphs.

Define a derivation D_1 on \mathcal{H}_L by

$$D_1(\Gamma_m) = \sum_{n=0}^{\infty} \Gamma_n Z_{0,n}(\Gamma_m)$$

which takes care of $m \circ \Delta$: for any two characters ϕ_1, ϕ_2 on \mathcal{H}_L we have

$$\phi_1 \star \phi_2(\Gamma_m) = m \circ (\phi_1 \otimes \phi_2) \circ \Delta(\Gamma_m) = \sum_{n=0}^{\infty} \phi_1(\Gamma_n) \phi_2(Z_{0,n}(\Gamma_m)).$$

The degree operator Y can be described by a derivation D_2 ,

$$D_2(\Gamma_m) = \sum_{k=1}^{\infty} Z_{k,k}(\Gamma_m) = m\Gamma_m$$

and $S(\Gamma_m)$ iteratively by a derivation D_3 ,

$$D_3(\Gamma_m) = -Z_{0,0}(\Gamma_m) - \sum_{n=0}^{\infty} D_3(\Gamma_n)Z_{1,n+1}(\Gamma_m).$$

Note that $Z_{k,k} = [Z_{k,0}, Z_{0,k}] + Z_{0,0}$, similarly $Z_{1,n+1}$ involves the commutator $[Z_{1,0}, Z_{0,n+1}]$.

Now we get $S \star Y$ from the definitions.

Proposition 5.1. *$S \star Y$ is the derivation that is uniquely given on the linear generators Γ_i as:*

$$S \star Y(\Gamma_m) = \sum_{n=0}^{\infty} D_3(\Gamma_n)D_2(Z_{0,n}(\Gamma_m)). \quad (16)$$

Remark 5.2. *Note that (16) holds as the coproduct is linear on generators on the lhs in the ladder case, and on the linear subspace of generators the antipode can indeed be described by a derivation.*

6. THE STANDARD MODULE Λ .

In this section we are going to address the following question: what is the equivalent of the standard module \mathcal{S} for the Lie algebras l^{\pm} ? Recall (Remark 2.5) that on $\mathcal{S} = \bigoplus_{n \geq 0} \mathbb{C}t_n$ one can naturally define the product:

$$\star : \mathcal{S} \otimes \mathcal{S} \longrightarrow \mathcal{S}$$

$$\star(t_n \otimes t_m) \mapsto t_{n+m}$$

and that with respect to this product \mathcal{S} is isomorphic to a polynomial algebra having only one generator. We also remember that the action of l^{\pm} is given in 4.2 (2). To define the standard module for l^{\pm} , we introduce the following notation:

$$o(n) \doteq -\exp(n - \frac{1}{2}), \quad n > 0;$$

$$e(n) \doteq \exp(n), \quad n \geq 0.$$

Definition 6.1. *Let us start defining the vector space:*

$$\Lambda = \text{span}_{\mathbb{C}}\{\alpha(o(n)), \alpha(e(n))\}$$

Λ is a unital algebra with the following product:

$$\bullet : \Lambda \otimes \Lambda \longrightarrow \Lambda$$

$$\alpha(\xi(n)) \bullet \alpha(\xi(m)) = \alpha(\xi(n)\xi(m));$$

where $\xi(k)$ can be either $o(k)$ or $e(k)$. The unit is given by $\alpha = \alpha(e(0))$.

Let us now define:

$$\phi : \mathcal{S} \longrightarrow \Lambda$$

by the following:

$$\phi(t_{2k}) = \alpha(e(k)), \quad k \geq 0 \quad \phi(t_{2k-1}) = \alpha(o(k)), \quad k > 0.$$

Proposition 6.2. *The map ϕ is an isomorphism of \mathbb{C} -algebras.*

PROOF We need only to check that ϕ is a morphism of algebras, for example:

$$\begin{aligned} \phi(t_{2n-1} \star t_{2m-1}) &= \phi(t_{2(n+m-1)}) = \\ \alpha(e(n+m-1)) &= \alpha(o(n)o(m)) = \\ \alpha(o(n)) \bullet \alpha(o(m)) &= \phi(t_{2n-1}) \bullet \phi(t_{2m-1}). \end{aligned}$$

□

We have now the following:

Proposition 6.3. *Λ is a l^{\pm} module.*

PROOF It suffices to define:

$$\lambda^{\pm} : l^{\pm} \longrightarrow \text{End}(\Lambda),$$

as multiplication operators since l^{\pm} are commutative Lie algebras and Λ is a commutative algebra. Let us define:

$$\begin{aligned} \lambda^+(Z_n^+)(\alpha(\xi(k))) &\doteq \alpha(e(n)) \bullet \alpha(\xi(k)), \\ \lambda^+(Z_{-n}^+)(\alpha(\xi(k))) &\doteq \alpha(o(n)) \bullet \alpha(\xi(k)) \end{aligned}$$

and:

$$\lambda^-(Z_n^-)(\alpha(\xi(k))) \doteq \alpha(\tilde{e}(n)) \bullet \alpha(\xi(k)), \text{ if } k - n \geq 0$$

and $\lambda^-(Z_n^-)(\alpha(\xi(k))) = 0$ otherwise;

$$\lambda^-(Z_{-n}^-)(\alpha(\xi(k))) \doteq \alpha(\tilde{o}(n)) \bullet \alpha(\xi(k)), \text{ if } k - n \geq 0,$$

and $\lambda^-(Z_{-n}^-)(\alpha(\xi(k))) = 0$ otherwise, where $\tilde{e}(n) = e(-n) = \exp(-n)$ while $\tilde{o}(n) = -\exp(-n + \frac{1}{2})$. \square

We can now state and prove the main result of this section:

Theorem 6.4. *The following diagrams commute:*

$$\begin{array}{ccc} S & \xrightarrow{Z_{2n,0}} & S \\ \downarrow \phi & & \downarrow \phi \\ \Lambda & \xrightarrow{Z_n^+} & \Lambda \end{array}$$

$$\begin{array}{ccc} S & \xrightarrow{Z_{2n-1,0}} & S \\ \downarrow \phi & & \downarrow \phi \\ \Lambda & \xrightarrow{Z_{-n}^+} & \Lambda \end{array}$$

The same statement holds for:

$$\begin{array}{ccc} S & \xrightarrow{Z_{0,2n}} & S \\ \downarrow \phi & & \downarrow \phi \\ \Lambda & \xrightarrow{Z_n^-} & \Lambda \end{array}$$

$$\begin{array}{ccc} S & \xrightarrow{Z_{0,2n-1}} & S \\ \downarrow \phi & & \downarrow \phi \\ \Lambda & \xrightarrow{Z_{-n}^-} & \Lambda \end{array}$$

PROOF It follows by inspection. Let us consider for example the fourth diagram:

$$\phi(Z_{0,2n-1}(t_{2k-1})) = \phi(t_{2(k-n)}) = \alpha(e(k-n))$$

and

$$\begin{aligned} Z_{-n}^-(\phi(t_{2k-1})) &= Z_{-n}^-(\alpha(-\exp(k - \frac{1}{2}))) = \alpha(-\exp(-n + \frac{1}{2})) \bullet \alpha(-\exp(k - \frac{1}{2})) = \\ &= \alpha(e(k-n)). \end{aligned}$$

Similarly:

$$\phi(Z_{0,2n-1}(t_{2k})) = \phi(t_{2(k-n)+1}) = \alpha(-\exp(k - n + \frac{1}{2})) = \alpha(o(k-n+1)),$$

and

$$Z_{-n}^-(\phi(t_{2k})) = Z_{-n}^-(\alpha(\exp(k))) = \alpha(-\exp(-n + \frac{1}{2})) \bullet \alpha(\exp(k)) =$$

$$= \alpha(-\exp(k - n + \frac{1}{2})) = \alpha(o(k - n + 1)).$$

□

7. CONCLUSIONS AND OUTLOOKS

With this paper we started the study of the Lie algebra introduced in [4]. We considered the simplest case for this Insertion-Elimination Lie algebra, the one coming from the sub-Hopf algebra of ladder graphs, giving a description of such a Lie algebra in terms of (infinite) matrices. We then described the relations of such Lie algebra with other well known infinite dimensional Lie algebras like the Heisenberg algebra and $\mathfrak{gl}(\infty)$.

In forthcoming works we will study the structure and representation theory of \mathcal{L}_L and we will consider the case of the Insertion-Elimination algebra coming from the ladder Hopf algebra with additional decorations, which makes the underlying Hopf algebra non-cocommutative, in contrast to the case studied here.

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REFERENCES

- [1] D. J. Broadhurst, D. Kreimer *Exact solutions of Dyson-Schwinger equations for iterated one-loop integrals and propagator-coupling duality*, Nucl. Phys. B **600**, 403 (2001) [arXiv:hep-th/0012146].
- [2] A. Connes, D. Kreimer *Renormalization in quantum field theory and the Riemann Hilbert problem. I. The Hopf algebra structure of graphs and the amini theorem*, Comm. Math. Phys. **210** (2000), no.1, 249-273.
- [3] A. Connes, D. Kreimer *Renormalization in quantum field theory and the Riemann Hilbert problem . II. The β -function, diffeomorphism and the renormalization group*, Comm. Math. Phys. **216** (2001), no.1, 215-241.
- [4] A. Connes, D. Kreimer, *Insertion and Elimination: the doubly infinite Lie algebra of Feynmann graphs*, Ann. Henri Poincare **3**, (2002) no. 3, 411-433.
- [5] V. Kac *Infinite Dimensional Lie Algebras*, Cambridge University Press, 3rd Ed. (1994).
- [6] V. Kac, A. Raina *Bombay Lectures on Highest Weight Representation of Infinite Dimensional Lie Algebras*, Advanced Series in Mathematical Physics, Vol 2, World Scientific Pub., 1988.

- [7] D. Kreimer *On the Hopf algebra structure of perturbative quantum field theory*, Adv. Theor. Math. Phys, **2**, no. 2, 1998.
- [8] J. W. Milnor, J. C. Moore *On the Structure of Hopf Algebras*, Ann. of Math, **81**, no. 2, 1965, 211-264.

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